

CALCUL INTÉGRAL

Intégration par décomposition en fractions rationnelles simples

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Calcule :

$$(1) \int \frac{x^2}{1-x^2} dx$$

$$\begin{array}{r} x^2 \\ -(x^2-1) \\ \hline 1 \end{array} \quad \left| \begin{array}{r} -x^2+1 \\ -1 \end{array} \right.$$

$$= \int \left(-1 + \frac{1}{1-x^2} \right) dx$$

$$= -x + \int \frac{1}{1-x^2} dx$$

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$= -x + \int \left(\frac{1/2}{1-x} + \frac{1/2}{1+x} \right) dx$$

$$\Leftrightarrow 1 = A \cdot (1+x) + B \cdot (1-x)$$

$$\Leftrightarrow 1 = A + Ax + B - Bx$$

$$= -x - \frac{1}{2} \ln|1-x| + \frac{1}{2} \ln|1+x| + C$$

$$\Leftrightarrow \begin{cases} 0 = A - B \rightarrow A = B \\ 1 = A + B \rightarrow 1 = 2A \\ A = \frac{1}{2} \rightarrow B = \frac{1}{2} \end{cases}$$

$$(2) \int \frac{x^3}{x+1} dx$$

$$\begin{array}{r} x^3 \\ -(x^3+x^2) \\ \hline -x^2 \\ -(-x^2-x) \\ \hline x \\ -(x+1) \\ \hline -1 \end{array} \quad \left| \begin{array}{r} x+1 \\ x^2-x+1 \end{array} \right.$$

$$= \int \left(x^2 - x + 1 - \frac{1}{x+1} \right) dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1| + C$$

$$(3) \int \frac{(1+x)^2}{1+x^2} dx = \int \frac{x^2+2x+1}{x^2+1} dx$$

$$= \int \left(1 + \frac{2x}{x^2+1} \right) dx$$

$$= x + \ln|x^2+1| + C$$

$$\begin{array}{r|l} x^2+2x+1 & x^2+1 \\ -(x^2+1) & 1 \\ \hline 2x & \end{array}$$

$$(4) \int \frac{1}{(x^2-4)(x^2+3)} dx = \int \frac{1}{(x-2)(x+2)(x^2+3)} dx$$

$$\frac{1}{(x-2)(x+2)(x^2+3)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+3}$$

$$\Leftrightarrow 1 = A(x+2)(x^2+3) + B(x-2)(x^2+3) + (Cx+D)(x^2-4)$$

$$\Leftrightarrow 1 = A(x^3+2x^2+3x+6) + B(x^3-2x^2+3x-6) + Cx^3+Dx^2-4Cx-4D$$

$$\Leftrightarrow \begin{cases} 0 = A+B+C & \rightarrow A = -B-C \\ 0 = 2A-2B+D & \rightarrow 0 = -2B-2C-2B+D \\ 0 = 3A+3B-4C & \rightarrow 0 = -4B-2C+D \\ 1 = 6A-6B-4D & \rightarrow D = 4B+2C \end{cases}$$

$$0 = -3B-3C+3B-4C$$

$$0 = -7C$$

$$C = 0$$

$$1 = -6B - 6B - 16B$$

$$B = -\frac{1}{28}$$

$$\Rightarrow D = -\frac{1}{7} \text{ et } A = \frac{1}{28}$$

$$\Rightarrow \int \left(\frac{1/28}{x-2} + \frac{-1/28}{x+2} + \frac{-1/7}{x^2+3} \right) dx$$

$$\int \frac{1}{x^2+3} dx = \frac{1}{\sqrt{3}} \int \frac{1 \cdot \frac{1}{\sqrt{3}}}{3 \cdot \left(1 + \left(\frac{x}{\sqrt{3}}\right)^2\right)} dx$$

$$= \frac{1}{28} \ln|x-2| - \frac{1}{28} \ln|x+2| - \frac{\sqrt{3}}{21} \arctan\left(\frac{\sqrt{3}x}{3}\right) + C$$

$$x^3 - 3x + 2 = (x-1)(x^2 + x - 2) \rightarrow \Delta = 9$$

$$= (x-1)(x-1)(x+2) \quad \alpha = \frac{-1 \pm 3}{2} \rightarrow \begin{array}{c|ccc|c} 1 & 0 & -3 & 2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

$$= (x-1)^2 \cdot (x+2)$$

$$(5) \int \frac{x^2}{(x^3 - 3x + 2)} dx$$

$$= \int \frac{x^2}{(x-1)^2(x+2)} dx$$

$$\frac{x^2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$\Leftrightarrow x^2 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$\Leftrightarrow x^2 = A(x^2 + x - 2) + B(x+2) + C(x^2 - 2x + 1)$$

$$\Leftrightarrow \begin{cases} 1 = A + C & \rightarrow C = 1 - A \\ 0 = A + B - 2C & \rightarrow 0 = A + B - 2 + 2A \\ 0 = -2A + 2B + C & B = 2 - 3A \end{cases}$$

$$\hookrightarrow 0 = -2A + 4 - 6A + 1 - A$$

$$A = \frac{5}{9} \Rightarrow C = \frac{4}{9} \text{ et } B = \frac{1}{3}$$

$$= \int \left(\frac{5/9}{x-1} + \frac{1/3}{(x-1)^2} + \frac{4/9}{x+2} \right) dx$$

$$= \frac{5}{9} \ln|x-1| + \frac{1}{3(x-1)} + \frac{4}{9} \ln|x+2| + C$$

$$(6) \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx$$

$$= \int \left(2x + \frac{5x-3}{x^2-2x-3} \right) dx$$

$$= x^2 + \int \frac{5x-3}{(x-3)(x+1)} dx$$

$$= x^2 + \int \left(\frac{3}{x-3} + \frac{2}{x+1} \right) dx$$

$$= x^2 + 3 \ln|x-3| + 2 \ln|x+1| + C$$

$$\begin{array}{r|l} 2x^3 - 4x^2 - x - 3 & x^2 - 2x - 3 \\ -(2x^3 - 4x^2 - 6x) & 2x \\ \hline 5x - 3 & \end{array}$$

$$\text{Racines: } x^2 - 2x - 3 = 0$$

$$\Delta = 4 + 12 = 16$$

$$x = \frac{2 \pm 4}{2} = \begin{matrix} 3 \\ -1 \end{matrix}$$

$$\frac{5x-3}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$\Leftrightarrow 5x-3 = A(x+1) + B(x-3)$$

$$\Leftrightarrow 5x-3 = Ax + A + Bx - 3B$$

$$\Leftrightarrow \begin{cases} 5 = A + B & \rightarrow A = 5 - B \\ -3 = A - 3B & \rightarrow -3 = 5 - B - 3B \\ & -8 = -4B \\ & B = 2 \Rightarrow A = 3 \end{cases}$$

$$(7) \int \frac{5x^3 - 3}{x(x-2)} dx = \int \frac{5x^3 - 3}{x^2 - 2x} dx$$

$$= \int \left(5x + 10 + \frac{20x - 3}{x \cdot (x-2)} \right) dx$$

$$= \frac{5x^2}{2} + 10x + \int \frac{20x - 3}{x \cdot (x-2)} dx$$

$$= \frac{5x^2}{2} + 10x + \int \left(\frac{20}{x} + \frac{37}{x-2} \right) dx$$

$$= \frac{5x^2}{2} + 10x + 20 \cdot \ln|x| + 37 \cdot \ln|x-2| + C$$

$$\begin{array}{r|l} 5x^3 - 3 & x^2 - 2x \\ \hline -(5x^3 - 10x^2) & 5x + 10 \\ \hline 10x^2 - 3 & \\ -(10x^2 - 20x) & \\ \hline 20x - 3 & \end{array}$$

$$\frac{20x - 3}{x \cdot (x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$$\Leftrightarrow 20x - 3 = A \cdot (x-2) + B \cdot x$$

$$\Leftrightarrow 20x - 3 = Ax - 2A + Bx$$

$$\Leftrightarrow \begin{cases} 20 = A \\ -3 = -2A + B \end{cases} \rightarrow \begin{cases} 20 = A \\ -3 = -40 + B \end{cases} \rightarrow B = 37$$

$$(8) \int \frac{5x^2}{(1+x)(4+x^2)} dx$$

$$= \int \left(\frac{1}{1+x} + \frac{4x-4}{4+x^2} \right) dx$$

$$= \int \left(\frac{1}{1+x} \right) dx + 4 \cdot \frac{1}{2} \int \frac{2x}{4+x^2} dx - 4 \int \frac{1}{4+x^2} dx$$

$$= \ln|1+x| + 2 \cdot \ln|4+x^2| - 4 \cdot \int \frac{1 \cdot \frac{1}{2}}{4 \cdot \left(1 + \left(\frac{x}{2}\right)^2\right)} dx$$

$$= \ln|1+x| + 2 \cdot \ln|4+x^2| - 2 \cdot \arctan\left(\frac{x}{2}\right) + C$$

$$\frac{5x^2}{(1+x)(4+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{4+x^2}$$

$$\Leftrightarrow 5x^2 = A \cdot (4+x^2) + (Bx+C) \cdot (1+x)$$

$$\Leftrightarrow 5x^2 = 4A + Ax^2 + Bx + Bx^2 + C + Cx$$

$$\Leftrightarrow \begin{cases} 5 = A+B \\ 0 = B+C \rightarrow B = -C \\ 0 = 4A+C \rightarrow A = -\frac{C}{4} \end{cases} \rightarrow \begin{cases} 5 = -\frac{C}{4} - C \rightarrow 5 = -\frac{5C}{4} \\ \Leftrightarrow C = -4 \end{cases}$$

$$\Rightarrow A = 1 \text{ et } B = 4$$