CALCUL INTÉGRAL

Intégration par substitution

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https://bit.ly/3ZF3ThE

QVC

Calcule :

(1) $\int x^3.\sqrt{1+3x^2}\,dx$

$u = 1+3x^2 \Leftrightarrow x = \left(\frac{u-1}{3}\right)^{1/2}$

$dx = \frac{1}{2}\cdot\frac{1}{3}\cdot\left(\frac{u-1}{3}\right)^{-1/2} du = \frac{1}{6}\cdot\left(\frac{u-1}{3}\right)^{-1/2} du$

$= \int\left(\left(\frac{u-1}{3}\right)^{1/2}\right)^3 \cdot u^{1/2}\cdot\frac{1}{6}\cdot\left(\frac{u-1}{3}\right)^{-1/2} du$

$= \frac{1}{6}\int\left(\frac{u-1}{3}\right)^{3/2}\cdot u^{1/2}\cdot\left(\frac{u-1}{3}\right)^{-1/2} du$

$= \frac{1}{6}\int \frac{u-1}{3}\cdot u^{1/2}\,du$

$= \frac{1}{18}\int\left(u^{3/2} - u^{1/2}\right) du$

$= \frac{1}{18}\left(\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right) + C$

$= \frac{1}{45}\sqrt{u^5} - \frac{1}{27}\sqrt{u^3} + C$

$= \frac{1}{45}\sqrt{(1+3x^2)^5} - \frac{1}{27}\sqrt{(1+3x^2)^3} + C$

(2) $\int \frac{x}{(1+x)^3}\,dx$

$u = 1+x \Leftrightarrow x = u-1$

$dx = du$

$= \int \frac{u-1}{u^3}\,du$

$= \int\left(\frac{1}{u^2} - \frac{1}{u^3}\right) du$

$= \int\left(\frac{1}{u^2} - u^{-3}\right) du$

$= \frac{-1}{u} - \frac{u^{-2}}{-2} + C$

$= \frac{-1}{u} + \frac{1}{2u^2} + C = \frac{-1}{1+x} + \frac{1}{2.(1+x)^2} + C$

(3) $\int \frac{x}{\sqrt{x+4}}\,dx$

$u = x+4 \Leftrightarrow x = u-4$

$dx = du$

$= \int \frac{u-4}{\sqrt{u}}\,du$

$= \int\left(u^{1/2} - 4.u^{-1/2}\right) du$

$= \frac{u^{3/2}}{3/2} - 4\cdot\frac{u^{1/2}}{1/2} + C$

$= \frac{2}{3}\sqrt{u^3} - 8.\sqrt{u} + C$

$= \frac{2}{3}\sqrt{(x+4)^3} - 8.\sqrt{x+4} + C$

(4) $\int \frac{x^2}{\sqrt{9-x^2}}dx$ $\quad x = 3\sin u \xrightarrow{} u = \arcsin\frac{x}{3}$

$dx = 3\cos u\, du$

$\rightarrow \sqrt{9-x^2} = \sqrt{9-9\sin^2 u}$

$= \sqrt{9.(1-\sin^2 u)}$

$= \sqrt{9.\cos^2 u}$

$= 3.\cos u$

$= \int \frac{9\sin^2 u}{3.\cos u}.3\cos u\, du$

$= 9\int \left(\frac{1-\cos 2u}{2}\right)du$

$= \frac{9}{2}\left(\int 1\, du - \frac{1}{2}\int 2.\cos 2u\, du\right)$

$= \frac{9}{2}\left(u - \frac{1}{2}\sin 2u\right) + C$

$= \frac{9}{2}u - \frac{9}{4}\sin 2u + C$

$= \frac{9}{2}\arcsin\frac{x}{3} - \frac{9}{4}\sin\left(2.\arcsin\frac{x}{3}\right) + C$

(5) $\int \sqrt{3-2x}.x^2\, dx$ $\quad u = 3-2x \Leftrightarrow x = \frac{3-u}{2}$

$dx = -\frac{1}{2}du$

$= \int \sqrt{u}.\left(\frac{3-u}{2}\right)^2.\frac{-1}{2}du$

$= -\frac{1}{2}\int u^{1/2}.\frac{9-6u+u^2}{4}du$

$= -\frac{1}{8}\int \left(9u^{1/2} - 6u^{3/2} + u^{5/2}\right)du$

$= -\frac{1}{8}\left(9\frac{u^{3/2}}{3/2} - 6\frac{u^{5/2}}{5/2} + \frac{u^{7/2}}{7/2}\right) + C$

$= -\frac{1}{8}.\left(6\sqrt{u^3} - \frac{12}{5}\sqrt{u^5} + \frac{2}{7}\sqrt{u^7}\right) + C$

$= \frac{-3}{4}.\sqrt{(3-2x)^3} + \frac{3}{10}\sqrt{(3-2x)^5} - \frac{1}{28}\sqrt{(3-2x)^7} + C.$

(6) $\int \sqrt{3+x}.(x+1)^2\,dx$ $\quad u = 3+x \iff x = u-3$

$dx = du$

$= \int \sqrt{u}.\ (u-3+1)^2.\ du$

$= \int \sqrt{u}.\ (u-2)^2\,du$

$= \int u^{1/2}.\ (u^2-4u+4)\,du$

$= \int (u^{5/2} - 4u^{3/2} + 4u^{1/2})\,du$

$= \dfrac{u^{7/2}}{7/2} - 4.\dfrac{u^{5/2}}{5/2} + 4.\dfrac{u^{3/2}}{3/2} + C$

$= \frac{2}{7}\sqrt{u^7} - \frac{8}{5}\sqrt{u^5} + \frac{8}{3}\sqrt{u^3} + C$

$= \frac{2}{7}\sqrt{(3+x)^7} - \frac{8}{5}\sqrt{(3+x)^5} + \frac{8}{3}\sqrt{(3+x)^3} + C$

(7) $\int \dfrac{dx}{\sqrt{2x-x^2}}$ (poser $u = x-1$) $\iff x = u+1$

$dx = du$

$= \int \dfrac{du}{\sqrt{2(u+1)-(u+1)^2}}$

$= \int \dfrac{du}{\sqrt{2u+2-(u^2+2u+1)}}$

$= \int \dfrac{du}{\sqrt{1-u^2}}$

$= \arcsin u + C$

$= \arcsin(x-1) + C$

(8) $\int \frac{e^x}{25+e^{2x}}\,dx$ $\qquad u = e^x \iff x = \ln u$

$dx = \frac{1}{u}\,du$

$= \int \frac{u}{25+u^2} \cdot \frac{1}{u}\,du$

$= \int \frac{1}{25+u^2}\,du$

$= \int \frac{1}{25\left(1+\frac{u^2}{25}\right)}\,du$

$= \frac{1}{25} \cdot 5 \int \frac{1 \cdot \frac{1}{5}}{1+\left(\frac{u}{5}\right)^2}\,du$

$= \frac{1}{5} \arctan\left(\frac{u}{5}\right) + C$

$= \frac{1}{5} \arctan\left(\frac{e^x}{5}\right) + C$

(9) $\int x^2\sqrt{1-x}\,dx$ $\qquad u = 1-x \iff x = 1-u$

$dx = -du$

$= \int (1-u)^2 \cdot \sqrt{u} \cdot (-du)$

$= -\int (1-2u+u^2) \cdot u^{1/2}\,du$

$= -\int (u^{1/2} - 2u^{3/2} + u^{5/2})\,du$

$= -\left(\frac{u^{3/2}}{3/2} - 2 \cdot \frac{u^{5/2}}{5/2} + \frac{u^{7/2}}{7/2}\right) + C$

$= -\frac{2}{3}\sqrt{u^3} + \frac{4}{5}\sqrt{u^5} - \frac{2}{7}\sqrt{u^7} + C$

$= -\frac{2}{3}\sqrt{(1-x)^3} + \frac{4}{5}\sqrt{(1-x)^5} - \frac{2}{7}\sqrt{(1-x)^7} + C$